MOMENTUM OPERATORS IN THE UNIT SQUARE

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Abstract. We investigate the skew-adjoint extensions of a partial derivative operator acting in the direction of one of the sides a unit square. We investigate the unitary equivalence of such extensions and the spectra of such extensions. It follows from our results, that such extensions need not have discete spectrum. We apply our techniques to the problem of finding commuting skew-adjoint extensions of the partial derivative operators acting in the directions of the sides of the unit square.

While our results are most easily stated for the unit square, they are established for a larger class of domains, including certain fractal domains.

Contents

1. Introduction

Consider $P_{\min} := \frac{1}{i2\pi}\frac{d}{dx}$ acting in $C_c^\infty([0,1])$. This operator is symmetric and its selfadjoint extensions are in one-to-one correspondence with the complex numbers $e(\theta) := e^{i2\pi\theta}$, $0 \leq \theta < 1$. The selfadjoint extension P_θ corresponding to $e(\theta)$ has domain

$$\operatorname{dom}(P_\theta) := \left\{ f \in L^2([0,1]) \mid f' \in L^2([0,1]), f(1) = e(\theta) f(0) \right\}$$

and $P_\theta f = \frac{1}{i2\pi} f'$, for f in $\operatorname{dom}(P_\theta)$, the derivative is in the distribution sense. The spectrum of P_θ is the set $\theta + \mathbb{Z} := \{\theta + m \mid m \in \mathbb{Z}\}$. See, for example, [RS75]. In particular, P_θ is unitary equivalent to $P_\theta + 1$ and P_θ is not unitary equivalent

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to $P_{\theta'}$ unless $\theta = \theta'$. In this paper we extend this analysis to $\frac{1}{i2\pi}\frac{\partial}{\partial x}$ acting in the unit square $[0,1]^2$. We apply our techniques to investigate the characterization of commuting selfadjoint realizations (extensions) of $\frac{1}{i2\pi}\frac{\partial}{\partial x}$ and $\frac{1}{i2\pi}\frac{\partial}{\partial y}$ in the infinite strip $[0,1]\times\mathbb{R}$, the square $[0,1]^2$, and in $[0,1]\times C$, for certain fractal sets C.

More precisely, we consider operators in the Hilbert space $L^2\left([0,1]^2\right)$ of square integrable functions $[0,1]^2 \to \mathbb{C}$ equipped with the inner product

$$\langle f,g\rangle = \int_0^1\int_0^1 \overline{f(x,y)}g(x,y)dxdy.$$

The operator $P_{\min} = \frac{1}{i2\pi}\partial_x$ with domain $\operatorname{dom}(P_{\min}) = C_c^\infty\left([0,1]^2\right)$ is *symmetric*, that is

$$\langle P_{\min} f, g\rangle = \langle f, P_{\min} g\rangle$$

for all f,g in $C_c^\infty\left([0,1]^2\right)$. The adjoint of $P_{\min}$ is $P_{\min}^* = P_{\max} = \frac{1}{i2\pi}\partial_x$, acting in the distribution sense, with domain

$$\operatorname{dom}(P_{\max}) = \left\{ f \in L^2\left([0,1]^2\right) \mid \partial_x f \in L^2\left([0,1]^2\right)\right\}.$$

Hence, for $k=0,1$, the map $f \to f(k,\cdot)$ maps $\operatorname{dom}(P_{\max})$ onto $L^2([0,1])$ and

$$\langle P_{\max} f, g\rangle - \langle f, P_{\max} g\rangle = i2\pi\left(\int_0^1 \overline{f(1,y)}g(1,y)dy - \int_0^1 \overline{f(0,y)}g(0,y)dy\right),$$

for all f and g in the domain of $P_{\max}$. The selfadjoint extensions of $P_{\min}$ are in one-to-one correspondence with the unitary operators V acting in $L^2([0,1])$: The selfadjoint extension P_V of $P_{\min}$ corresponding to the unitary V is the restriction of $P_{\max}$ whose domain $\operatorname{dom}(P_V)$ is the functions f in $\operatorname{dom}(P_{\max})$ that satisfies the boundary condition

$$f(1,\cdot) = Vf(0,\cdot). \tag{1.1}$$

We will call V a *boundary* unitary.

In the interval case P_θ and $P_{\theta'}$ are unitary equivalent if and only if $\theta=\theta'$. The analogue for the unit square is:

Theorem 1.1. *Let $U,V : L^2([0,1]) \to L^2([0,1])$ be two boundary unitary operators. Then the corresponding selfadjoint extensions P_U and P_V of $P_{\min}$ are unitary equivalent if and only if U and V are unitary equivalent.*

In the case of the interval the spectrum P_θ is discrete, in fact equal to $\theta+\mathbb{Z}$. But for the square the spectrum of P_V has a much richer structure. Theorem 4.3 gives a description of the spectral measure associated with P_V in terms of the spectral measure associated with V. Theorems 1.2 and 1.3 are consequences of Theorem 4.3.

Theorem 1.2. *Let P_V be a selfadjoint extension of $P_{\min}$ associated with the boundary unitary operator $V : L^2([0,1]) \to L^2([0,1])$. Then $P_V + 1$ is unitary equivalent to P_V.*

A consequence of the following result is that, in contrast to the interval case, the spectrum of P_V need not be discrete.

Theorem 1.3. *The spectrum of P_V equals the set of λ for which $e(\lambda)$ is in the spectrum of V.*

A pair (μ, ν) of measures on $\mathbb{R}^d$ is called a *spectral pair*, if $F : f \to \widehat{f}(\lambda) := \int f(x) e(-\lambda x) \, d\mu(x)$ determines a unitary $F : L^2(\mu) \to L^2(\nu)$. In this form the notion was introduced in [JP99]. The case where μ is a the restriction of Lebesgue measure to a measurable set Ω was studied in [Ped87], in this case the set Ω is called a *spectral set*, provided (μ, ν) is a spectral pair for some measure ν. The notion was introduced in [Fug74] in the case where Ω has finite Lebesgue measure. A connected open set in $\mathbb{R}^d$ is a spectral set if and only if there are commuting selfadjoint extensions of the partial derivatives $\frac{1}{i2\pi} \partial_{x_k}\Big|_{C_c^\infty(\Omega)}$, $k = 1, \ldots, d$, in $L^2(\Omega)$. In the affirmative case, the support of ν is the joint spectrum of the commuting selfadjoint extensions. See, [Fug74], [Jør82], and [Ped87] for proofs of these claims. We use Theorem 4.3 to characterize the boundary unitary operators that lead to commuting extensions of the partial derivatives and we calculate the joint spectra for the infinite strip, the unit square, and a fractal domain. This was previously done for the unit square, by a different method, in [JP00].

While our primary interest is in the unit square, we find it convenient to establish many of our results in a more abstract setting. This also allows us to apply our techniques to the study of spectral sets. Using

$$L^2\left([0,1]^2\right) = L^2([0,1]) \otimes L^2([0,1]),$$

we replace the second L^2-space by a generic Hilbert space and we replace the interval $[0,1]$ by the generic interval $[\alpha, \beta]$.

Fix real numbers $\alpha < \beta$ and a Hilbert space H. Consider the Hilbert space

$$\mathscr{H} := L^2([\alpha, \beta], H) = L^2([\alpha, \beta]) \otimes H$$

of L^2-functions $[\alpha, \beta] \to H$ equipped with the inner product

$$\langle f \mid g \rangle := \int_\alpha^\beta \langle f(x) \mid g(x) \rangle \, dx,$$

where $\langle f(x) \mid g(x) \rangle$ is the inner product in H. We will consider selfadjoint extensions of the operator P_0 determined by

$$P_0 f := \tfrac{1}{i2\pi} f'$$

with the domain

$$\operatorname{dom}(P_0) := \left\{ f \in L^2([\alpha, \beta], H) \mid f' \in L^2([\alpha, \beta], H), f(\alpha) = f(\beta) = 0 \right\}.$$

The selfadjoint extensions of P_0 are determined by boundary conditions, more precisely, they are parametrized by the unitary operators $U : H \to H$. The selfadjoint extension P_U corresponding to the unitary U is

$$P_U f = \tfrac{1}{i2\pi} f' \tag{1.2}$$

for

$$f \in \operatorname{dom}(P_U) := \left\{ f \in L^2([\alpha, \beta], H) \mid f' \in L^2([\alpha, \beta], H), f(\beta) = U f(\alpha) \right\}. \tag{1.3}$$

For more details on this correspondence, see Appendix A.

Section 2 contains a formula for the unitary group $e(aP_U)$. This formula is used to prove Theorem 1.1. In Section 3 we discuss eigenvalues and eigenvectors of P_U and we present some natural examples where P_U has a complete set of eigenvectors. In Section 4 we establish the connection, Theorem 4.3, between the projection-valued measures of P_U and of U. We use this connection to establish Theorems 1.2

and 1.3. Section 5 includes the analysis of spectral pairs discussed above. Appendix A contains the details needed to establish that the collection P_U, U unitary in H is the collection of all self adjoint extensions of P_0. Appendix B contains some open problems.

The papers [Hir00] and [Ree88] contains discussions of momentum operators in the complements of simple compact sets. In fact, any paper discussing the canonical commutation relations in proper subsets of $\mathbb{R}^d$, for $d > 1$ contains, at least implicitly, material related to the present paper. For other recent work on momentum operators we refer to [Car99], [ES10], [Exn12], [FKW07], [JPT12a], [JPT12d],[JPT12b], and [JPT12c].

This paper is based on standard operator theory, the needed background can be found in [RS72, RS75]. Some recent text books containing most, but not all, of what we need are [dO09] and [Gru09].

2. The Unitary Group

Fix $\alpha < \beta$. For a real number r, let $\alpha \le \langle r \rangle < \beta$ and $\lfloor r \rfloor \in \mathbb{Z}$ be such that $r = \langle r \rangle + \lfloor r \rfloor (\beta - \alpha)$. Note, $\langle r \rangle$ and $\lfloor r \rfloor$ are uniquely determined by these conditions. For a fixed real number a, the transformation $\tau_a : x \to \langle x + a \rangle$ is a measure preserving transformation of $[\alpha, \beta]$, hence $T_a f := f \circ \tau_a$ is a unitary in $L^2([\alpha, \beta])$.

Corresponding to any selfadjoint extension P_V, there is strongly continuous unitary one-parameter group $a \to e(aP_V)$, where

$$e(r) := e^{i2\pi r}.$$

The unitary group can, for example, be determined from P_V by an application of the spectral theorem. The result below establishes an explicit formula for the action of $e(aP_V)$. Conversely, P_V can be obtained from $e(aP_V)$ by differentiating $a \to e(aP_V)$ at $a = 0$.

Proposition 2.1. *Let $V : H \to H$ be a unitary and let P_V be the corresponding selfadjoint extension of P_0 determined by (1.2) and (1.3). The unitary group $a \to e(aP_V)$ satisfies*

$$e(aP_V) f(x) = \left(T_a \otimes V^{\lfloor x+a \rfloor} \right) f(x) = V^{\lfloor x+a \rfloor} f(\langle x + a \rangle) \tag{2.1}$$

for all f in $L^2([\alpha, \beta]) \otimes H$, a.e. x in $[\alpha, \beta]$, and all a in $\mathbb{R}$.

Proof. This is well know, we include a proof for completeness. Let $U_a := T_a \otimes V^{\lfloor x+a \rfloor}$. We must show that $e(aP_V) = U_a$. We begin by checking that U_a is a strongly continuous unitary one-parameter group.

Let I denote the identity in $L^2([\alpha, \beta])$. Then $I \otimes V^{\lfloor x+a \rfloor}$ is unitary because

$$I \otimes V^{\lfloor x+a \rfloor} = \begin{pmatrix} I \otimes V^{\lfloor a \rfloor} & 0 \\ 0 & I \otimes V^{1+\lfloor a \rfloor} \end{pmatrix}$$

with respect to the decomposition

$$L^2([0,1]) \otimes H = \left(L^2([\alpha, \beta - \langle a \rangle) \otimes H \right) \oplus \left(L^2([\beta - \langle a \rangle, \beta) \otimes H \right)$$

of $L^2([0,1]) \otimes H$. Hence U_a is unitary, since T_a is unitary.

Next, we check that U_a, $a \in \mathbb{R}$ is a group action, that is that $U_a U_b = U_{a+b}$, for all a and b in $\mathbb{R}$. Consider $f(x) = g(x)h$, where $g \in L^2([\alpha, \beta])$ and $h \in H$, then

$$I \otimes V^{\lfloor x+a \rfloor} f(x) = g(x) V^{\lfloor x+a \rfloor} h.$$

Hence,

$$U_a f(x) = g(\langle x+a\rangle) V^{\lfloor x+a\rfloor} h$$

so

$$\begin{aligned} U_a (U_b f)(x) &= U_b \Big(g(\langle x+a\rangle) \Big(V^{\lfloor x+a\rfloor} h \Big) \Big) \\ &= g(\langle\langle x+a\rangle + b\rangle) V^{\lfloor \langle x+a\rangle + b\rfloor} \Big(V^{\lfloor x+a\rfloor} h \Big). \end{aligned}$$

Now $\langle\langle x+a\rangle+b\rangle = \langle x+a+b\rangle$ and $\lfloor\langle x+a\rangle+b\rfloor + \lfloor x+a\rfloor = \lfloor x+a+b\rfloor$, hence $U_aU_b = U_{a+b}$ for all simple tensors $f(x) = g(x)h$ and therefore for all f in $L^2([\alpha,\beta])\otimes H$.

To see that $a \to U_a f$ is continuous. Consider $f \in L^2([\alpha,\beta-b))\otimes H$ for some $0 < b < \beta-\alpha$. Then

$$U_a f(x) = f(x+a)$$

for all $a < b$. Hence $U_a f \to f$ as $a \to 0$. Since, $\bigcup_{0<b<\beta-\alpha} L^2([\alpha,\beta-b))\otimes H$ is dense in $L^2([\alpha,\beta])\otimes H$, we conclude that U_a is strongly continuous.

Since U_a is a strongly continuous unitary group on $L^2([\alpha,\beta])\otimes H$, there is a selfadjoint operator Q on $L^2([\alpha,\beta])\otimes H$ such that $e(aQ) = U_a$ for all $a \in \mathbb{R}$. Now

$$\tfrac{1}{i2\pi} Qf = \lim_{a\to 0} \tfrac{1}{i2\pi a}(U_a f - f)$$

and the domain of Q is the set of all f for which the limit exists. If $f : [\alpha,\beta] \to H$ is compactly supported in (α,β), then for sufficiently small a we have

$$\tfrac{1}{i2\pi a}(U_a f - f)(x) = \tfrac{1}{i2\pi a}(f(x+a) - f(x)),$$

by definition of U_a. Consequently, $P_0 \subset Q$, meaning that P_0 is a restriction of Q. Taking the adjoint gives $Q \subset P_0^*$. Hence, if $f \in \mathrm{dom}(Q)$, then $f \in \mathrm{dom}(P_0^*)$, hence, by Lemma A.1, $f'(x)$ exists and $Qf(x) = \frac{1}{i2\pi} f'(x)$ for a.e. x in (α,β). Let $0 < a < \beta-\alpha$. then we have

$$\tfrac{1}{i2\pi a}(U_a f - f)(x) = \tfrac{1}{i2\pi a}(Vf(\langle x+a\rangle) - f(x)) \tag{2.2}$$

when $\beta - a < x < \beta$, by definition of U_a. As $a \to 0$, we have $x \to \beta$, $\langle x+a\rangle \to \alpha$, so $f(x) \to f(\beta)$, and $Vf(\langle x+a\rangle) \to Vf(\alpha)$. Thus, (2.2) implies $f(\beta) = Vf(\alpha)$. It follows that $\mathrm{dom}(Q) \subseteq \mathrm{dom}(P_V)$. Consequently, $Q = P_V$ as we needed to show. □

Lemma 2.2. *The unitary group $e(aP_V)$ in $L^2([\alpha,\beta])\otimes H$ and the boundary unitary V in H are related by*

$$e((\beta-\alpha)P_V) = I \otimes V, \tag{2.3}$$

where I is the identity operator in $L^2([\alpha,\beta])$.

Proof. Set $a = \beta - \alpha$ in (2.1) and use that $T_{\beta-\alpha} = I$. □

Theorem 1.1 is a consequence of

Theorem 2.3. *Let $U, V : H \to H$ be two boundary unitary operators. The corresponding selfadjoint extensions P_U and P_V of P_0, in $L^2([\alpha,\beta])\otimes H$, determined by (1.2) and (1.3) are unitary equivalent if and only if U and V are unitary equivalent.*

Proof. Suppose P_U and P_V are unitary equivalent. Let W be a unitary such that $P_U = W^* P_V W$, then it follows from the spectral theorem that

$$We(aH_U) = e(aH_V)W \tag{2.4}$$

for all real numbers a. Setting $a = \beta - \alpha$ in (2.6) and using Lemma 2.2 leads to

$$W (I \otimes U) = (I \otimes V) W. \tag{2.5}$$

Where I is the identity operator acting in $L^2 ([\alpha, \beta])$. By Proposition 2.1, equation (2.4) takes the form

$$W \left(T_a \otimes U^{\lfloor x+a \rfloor}\right) = \left(T_a \otimes V^{\lfloor x+a \rfloor}\right) W \tag{2.6}$$

for all real numbers a. Combining (2.6) and (2.5) we have

$$\left(I \otimes V^{\lfloor x+a \rfloor}\right) W (T_a \otimes I_H) = \left(I \otimes V^{\lfloor x+a \rfloor}\right) (T_a \otimes I_H) W,$$

where I_H is the identity in H. Therefore

$$W (T_a \otimes I_H) = (T_a \otimes I_H) W. \tag{2.7}$$

Let $e_m(x) = e(mx)$ for $x, m \in \mathbb{R}$. Applying (2.7) to $f = e_m \otimes h$, $m \in \mathbb{R}$, $h \in H$, we get

$$e(ma) W (e_m \otimes h) (x) = (W (e_m \otimes h)) (\langle x + a \rangle)$$

for all a. Consequently, there are h_m in H such that

$$W (e_m \otimes h) = e_m \otimes h_m \tag{2.8}$$

for all m. Let P be the projection in $L^2 ([\alpha, \beta]) \otimes H$ onto the functions that are independent of x. Setting $m = 0$ in (2.8) shows that the range of P is invariant under W. Hence,

$$WP = PWP. \tag{2.9}$$

Taking the adjoint of (2.7) and repeating this argument shows that

$$W^* P = PW^* P. \tag{2.10}$$

Combining (2.9) and (2.10) we get

$$WP = PW \quad \text{and} \quad W^* P = PW^*. \tag{2.11}$$

Let $i : H \to L^2 ([\alpha, \beta]) \otimes H$ be the isometric embedding determined by

$$(ig) (x) = (\beta - \alpha)^{-1/2} g,$$

for all x in $[\alpha, \beta]$. Then

$$i^* f = (\beta - \alpha)^{-1/2} \int_\alpha^\beta f(x)\, dx$$

for $f \in L^2 ([\alpha, \beta]) \otimes H$. And, if $B : H \to H$ is a bounded operator, then

$$(I \otimes B) i = iB. \tag{2.12}$$

Replacing B by B^* in (2.12) and taking the adjoint yields

$$i^* (I \otimes B) = Bi^*. \tag{2.13}$$

Consequently,

$$\begin{aligned} (i^* W i) U &= i^* W (iU) \\ &= i^* W (I \otimes U) i && \text{(by (2.12))} \\ &= i^* (I \otimes V) W i && \text{(by (2.5))} \\ &= V (i^* W i). && \text{(by (2.13))} \end{aligned}$$

It remains to show i^*Wi is unitary. It is easy to see that $P = ii^*$ and $i^*i = I_H$. Recall, I_H is the identity in H.

Using (2.11), $i^*i = P$, and $Pi = i$ simple calculations show that

$$(i^*Wi)^* (i^*Wi) = I_H$$

and

$$(i^*Wi) (i^*Wi)^* = I_H.$$

Therefore, i^*Wi is unitary, and U is unitary equivalent to V.

Conversely, suppose U is unitary equivalent to V. Let W be a unitary in H such that $WU = VW$. Then, by Proposition 2.1

$$\begin{aligned}(I \otimes W)\, e(aP_U)\, f(x) &= (I \otimes W) \left(T_a \otimes U^{\lfloor x+a \rfloor}\right) f(x) \\ &= \left(T_a \otimes V^{\lfloor x+a \rfloor}\right) (I \otimes W)\, f(x) \\ &= e(aP_V)\,(I \otimes W)\, f(x),\end{aligned}$$

for all $a \in \mathbb{R}$. Consequently, $(I \otimes W) P_U = P_V (I \otimes W)$. □

3. Eigenvalues

The first result in this section establishes a relationship between the eigenvalues and eigenvectors of V and the eigenvalues and eigenvectors of P_V. We extend this result to include continuous spectrum in Section 4, see Theorem 4.3.

Proposition 3.1. *Let $V : H \to H$ be a unitary and let P_V be the corresponding selfadjoint extension of P_0, in $L^2([\alpha, \beta], H)$, determined by (1.2) and (1.3). Then λ is an eigenvalues of P_V if and only if $e((\beta - \alpha)\lambda)$ is an eigenvalue of V. In particular, if λ is an eigenvalue for P_V, so is $\lambda + \frac{m}{\beta - \alpha}$, for any integer m. Furthermore, h_j, $1 \le j < n+1$ is an orthogonal basis for the eigenspace of V corresponding to the eigenvalue $e((\beta - \alpha)\lambda)$ if and only if $f_j(x) = e(\lambda x)h_j$ is an orthogonal basis for the eigenspace of P_V corresponding to the eigenvalue λ.*

Proof. Suppose $e((\beta - \alpha)\lambda)$ is an eigenvalue of V and $h \in H$ is a corresponding eigenvector. Let

$$f(x) := e(\lambda x)h.$$

Then, $Vf(\alpha) = e(\lambda\alpha)Vh = e(\lambda\beta)h = f(\beta)$, hence f is in the domain of P_V. Since

$$P_V f(x) = \frac{1}{i2\pi}\partial_x f(x) = \frac{1}{i2\pi}\partial_x e(\lambda x)h = \lambda e(\lambda x)h = \lambda f(x)$$

we conclude that λ is an eigenvalue of P_V

Conversely, suppose λ is an eigenvalue for P_V and f is a corresponding eigenvector. Then $P_V f = \lambda f$ implies $\partial_x f = 2\pi i \lambda f$. Solving this differential equation gives $f(x) = e(\lambda x)h$ for some $h \in H$. Since f is in the domain of P_V, it follows from (1.3) that $Vh = e((\beta - \alpha)\lambda)h$. Consequently, $e((\beta - \alpha)\lambda)$ is an eigenvalue for V.

We leave the details of the eigenvector claims to the reader. □

In the remainder of this section we consider the case where $\alpha = 0$, $\beta = 1$, and $H = L^2([0,1])$. Hence $\mathscr{H} = L^2\left([0,1]^2\right)$. For a measure preserving transformation $v : [0,1] \to [0,1]$ and a measurable function $\theta : [0,1] \to \mathbb{R}$, let $V = V_{v,\theta}$ be the unitary on $L^2([0,1])$ determined by

$$Vg(y) = e(\theta(y))\, g(v(y)). \tag{3.1}$$

Let $v^0(y) = y$ and inductively $v^n = v \circ v^{n-1}$ for $n > 0$. If

$$\phi_a(x,y) = \left(\langle a+x\rangle, v^{\lfloor x+a\rfloor} y\right),$$

then it follows from (2.1) that

$$\begin{aligned} e(aP_V)f(x,y) &= e(\lfloor x+a\rfloor \theta(y)) f\left(\langle a+x\rangle, v^{\lfloor x+a\rfloor}(y)\right) \\ &= e(\lfloor x+a\rfloor \theta(y)) f\circ \phi_a(x,y) \end{aligned}$$

for f in $L^2\left([0,1]^2\right)$.

Remark 3.2 (Geometric Boundary Conditions)*.* In the case of the unit interval $[0,1]$, the selfadjoint momentum operators are determined by the boundary condition $f(1) = e(\theta)f(0)$. Geometrically, we can think of this as identifying the endpoints up to a phase shift. A natural analogue of this for the unit square is the special case $Vg(y) = e(\theta)g(\langle y+r\rangle)$, $r, \theta \in \mathbb{R}$, of (3.1), in this case the spectrum of V is well understood, see Example 3.5. A more general analogue of the interval case is $Vg(y) = e(\theta(y))\, g(\langle y+r\rangle)$, for some measurable $\theta : [0,1] \to \mathbb{R}$. In this case, the spectral type of V is pure and the multiplicity is uniform, see [Hel86]. The exact spectral type depends on the function θ, see, for example, [ILM99] and the references therein.

We have the following corollary to Proposition 3.1.

Corollary 3.3. *v is an ergodic transformation on $[0,1]$ if and only if ϕ_a is an ergodic action of $\mathbb{R}$ on $[0,1]^2$.*

Proof. Let $\theta(y) = 0$ for all y. It follows from Proposition 3.1 that 1 is an eigenvalue for V with multiplicity one if and only if 1 is an eigenvalue for $e(aP_V)$ with multiplicity one. □

A special case of Proposition 3.1 is:

Corollary 3.4. *Fix r_n in $[0,1[$. Let V be determined by $Ve_n = e(r_n)e_n$, then the set*

$$\bigcup_{n\in\mathbb{Z}} (r_n + \mathbb{Z})$$

equals the set of eigenvalues for P_V.

Rotations provide a natural class of examples for the Corollary 3.4 and Corollary 3.3:

Example 3.5 (Rotations)**.** Let $0 \le r < 1$ be a real number. Consider

$$(Vg)(y) = g(v(y))$$

where $v(y) = \langle y+r\rangle$ is the fractional part of $y+r$. Using Fourier series,

$$g(y) = \sum_{n\in\mathbb{Z}} \widehat{g}(n)e(ny)$$

where $\widehat{g}(n) = \int_0^1 g(y)e(-ny)dy$, it follows that

$$V\sum_{n\in\mathbb{Z}} \widehat{g}(n)e(ny) = \sum_{n\in\mathbb{Z}} \widehat{g}(n)e(nr)e(ny).$$

In particular, $e_n(y) = e(ny)$ is an eigenfunction for V corresponding to the eigenvalue $e(nr)$. So, by Proposition 3.1, the set of eigenvalues for P_V is the set $r\mathbb{Z}+\mathbb{Z} = \{ra+b \mid a,b \in \mathbb{Z}\}$. Compared to the previous example we have $r_n = \langle nr \rangle$, the fractional part of nr. This is used in Remark 5.7.

If r is irrational, then v is ergodic. Clearly, $r_m \neq r_n$ for all $m \neq n$, so each eigenvalue for P_V has multiplicity one. Furthermore, it is well known that the sequence r_n is uniformly distributed in the interval $[0,1]$. See e.g., [KN74].

If r is rational, the set $\{r_n \mid n \in \mathbb{Z}\}$ is finite and each eigenvalue of P_V has infinite multiplicity.

Example 3.6. If $V_k g = g \circ v_k$, where $v_1(y) = 1-y$ and $v_2(y) = \tau_{1/2}(y) = \langle y + 1/2 \rangle$ is the fractional part of $y+1/2$, then V_1 and V_2 are unitary equivalent. Hence, P_{V_1} and P_{V_2} are unitary equivalent, by Theorem 2.3. However, dynamically v_1 is a reflection and v_2 is a translation.

4. A Spectral Theorem

In this section we obtain a formula for the spectral resolution P_V in terms of the the spectral resolution of the boundary unitary V. This is essentially contained in Section 3 for the set of eigenvalues. We begin working toward the spectral representation of P_V, when there is continuous spectrum, by finding the Green's function and using it to find a formula for the resolvent of P_V.

Proposition 4.1. *Consider a boundary unitary $V : H \to H$ and the corresponding selfadjoint extension P_V of P_0, in $L^2([\alpha,\beta]) \otimes H$, determined by (1.2) and (1.3). For all $f \in L^2([\alpha,\beta], H)$, and $z \in \mathbb{C}\backslash\mathbb{R}$,*

$$(z - P_V)^{-1} f(x,\cdot) = \int_\alpha^\beta G(x,s,z) f(s,\cdot)\,ds \tag{4.1}$$

where the Green's function $G(x,s,z)$, is given by

$$G(x,s,z) = \begin{cases} i2\pi\left(\left(1 - Ve_{\beta-\alpha}(-z)\right)^{-1} - 1\right) e(z(x-s)) & \alpha \leq s < x \leq \beta \\ i2\pi\left(1 - Ve_{\beta-\alpha}(-z)\right)^{-1} e(z(x-s)) & \alpha \leq x < s \leq \beta \end{cases} \tag{4.2}$$

for all $z \in \mathbb{C}\backslash\mathbb{R}$. Where $e_\lambda(z) := e(\lambda z) = e^{i2\pi\lambda z}$.

Proof. Let $f \in L^2([\alpha,\beta], H)$, $z \in \mathbb{C}\backslash\mathbb{R}$, and let

$$g := (z - P_V)^{-1} f \in \operatorname{dom}(P_V).$$

That is, g is the unique solution to the differential equation

$$zg(x) - \frac{1}{2\pi i} g'(x) = f(x) \tag{4.3}$$

satisfying the boundary condition

$$Vg(\alpha) = g(\beta). \tag{4.4}$$

Multiply both sides of (4.3) by the integrating factor $e(-zx)$, we get

$$\frac{d}{dx}\left(e(-zx)\,g(x)\right) = -i2\pi\, e(-zx) f(x)$$

so that

$$g(x) = e(z(x-\alpha))\,g(\alpha) - i2\pi \int_\alpha^x e(z(x-s)) f(s)\,ds. \tag{4.5}$$

By the boundary condition (4.4),

$$\left(V-e_{\beta-\alpha}\left(z\right)\right)g\left(\alpha\right)=-i2\pi\int_{\alpha}^{\beta}e\left(z\left(\beta-s\right)\right)f\left(s\right)ds.$$

Note that $g\left(\alpha\right)$ has a unique solution if and only if $e_{\beta-\alpha}\left(z\right)\notin sp\left(V\right)$. In that case,

$$g\left(\alpha\right)=i2\pi\left(e_{\beta-\alpha}\left(z\right)-V\right)^{-1}\int_{\alpha}^{\beta}e\left(z\left(\beta-s\right)\right)f\left(s\right)ds. \tag{4.6}$$

Substitute $g\left(\alpha\right)$ into (4.5), and it follows that

$$\begin{aligned}\left(z-P_{V}\right)^{-1}f\left(x\right) &= \int_{\alpha}^{x}i2\pi\left(\left(1-Ve_{\beta-\alpha}\left(-z\right)\right)^{-1}-1\right)e\left(z\left(x-s\right)\right)f\left(s\right)ds\\ &+ \int_{x}^{\beta}i2\pi\left(1-Ve_{\beta-\alpha}\left(-z\right)\right)^{-1}e\left(z\left(x-s\right)\right)f\left(s\right)ds.\end{aligned}$$

Eq. (4.2) follows from this. □

Remark 4.2 (Distribution Theory). Fix $z\in\mathbb{C}\backslash\mathbb{R}$, and $s\in\left(\alpha,\beta\right)$. The Green's function $G\left(\cdot,s,z\right)\in L^{2}\left(\left[\alpha,\beta\right],H\right)$ is the unique solution to the differential equation

$$zf-\frac{1}{i2\pi}f'=\delta_{s}. \tag{4.7}$$

Here, δ_{s} is the Dirac measure supported at $s\in\left(\alpha,\beta\right)$.

For $x\neq s$, G is the homogeneous solution, and so

$$G\left(x,s,z\right)=\begin{cases}c_{1}e\left(z\left(x-s\right)\right) & \alpha\leq x<s\leq\beta\\ c_{2}e\left(z\left(x-s\right)\right) & \alpha\leq s<x\leq\beta\end{cases} \tag{4.8}$$

where c_{1} and c_{2} are independent of x.

Moreover, from the theory of distributions, (4.7) implies that

$$G\left(s-,s,z\right)-G\left(s+,s,z\right)=i2\pi; \tag{4.9}$$

and by the boundary condition (4.4),

$$G\left(\beta,s,z\right)=VG\left(\alpha,s,z\right). \tag{4.10}$$

Combining (4.8), (4.9) and (4.10), we get

$$c_{1} = i2\pi\left(1-Ve_{\beta-\alpha}\left(-z\right)\right)^{-1} \tag{4.11}$$

$$c_{2} = i2\pi\left(\left(1-Ve_{\beta-\alpha}\left(-z\right)\right)^{-1}-1\right) \tag{4.12}$$

which in turn yields (4.2).

From the Green's function, one may reconstruct the projection-valued measure associated with P_{V}. Our formula for this measure involves the spectral resolution E_{V} in H of the boundary unitary, written in the form

$$V=\int_{[0,1)}e(\lambda)E_{V}(d\lambda) \tag{4.13}$$

and the projections

$$E_{\frac{\lambda+m}{\beta-\alpha}}f:=\left(\beta-\alpha\right)^{-1}\left(\int_{\alpha}^{\beta}f(t)e\left(\tfrac{\lambda+m}{\beta-\alpha}t\right)dt\right)e_{\frac{\lambda+m}{\beta-\alpha}} \tag{4.14}$$

in $L^2([\alpha,\beta])$, $m\in\mathbb{Z}$, onto the subspaces spanned by the unit vectors

$$f_{\lambda+m}(x) := (\beta-\alpha)^{-1/2}\, e\left(\tfrac{\lambda+m}{\beta-\alpha}x\right).$$

We use $[0,1)$ in the spectral resolution of V, to indicate that, if 1 is an eigenvalue of V, then the corresponding atom of E_V is located at 0 and not at 1. For any $\lambda\in\mathbb{R}$, the functions $f_{\lambda+m}, m\in\mathbb{Z}$ form an orthonormal basis for $L^2([\alpha,\beta])$ and $T_a f_n = e\left(\frac{na}{\beta-\alpha}\right) f_n$ for all $a\in\mathbb{R}$ and all $n\in\mathbb{Z}$.

Theorem 4.3. *Let P_V be the selfadjoint extension of P_0, in $L^2([\alpha,\beta])\otimes H$, associated with the boundary unitary operator $V: H\to H$ by (1.2) and (1.3). Suppose*

$$P_V \quad = \quad \int_{\mathbb{R}} \lambda F(d\lambda)$$

where $F(d\lambda)$ is the projection-valued measure of P_V. Then, for all $-\infty<\mu<\nu<+\infty$,

$$\tfrac{1}{2}\left(F((\mu,\nu)) + F([\mu,\nu])\right) = \int_{[0,1)} \sum_{m\in\mathbb{Z}} \chi_{(\mu,\nu)}\left(\tfrac{\lambda+m}{\beta-\alpha}\right) E_{\frac{\lambda+m}{\beta-\alpha}} \otimes E_V(d\lambda), \tag{4.15}$$

where $E_V(d\lambda)$ is as in (4.13) and $E_{\frac{\lambda+m}{\beta-\alpha}}$ is as in (4.14).

Proof. Let $-\infty<\mu<\nu<\infty$. By Stone's formula [RS72, pages 237 and 264] we have

$$\tfrac{1}{2}\left(F((\mu,\nu)) + F([\mu,\nu])\right) = \lim_{b\searrow 0}\int_{\mathbb{R}} \chi_{(\mu,\nu)}(a)\left((\overline{z}-P_V)^{-1} - (z-P_V)^{-1}\right) da, \tag{4.16}$$

where $z=z(a,b):=a+ib$. It follows from (4.5) and (4.6) that for $w\in\mathbb{C}\setminus\mathbb{R}$ and $f\in L^2([\alpha,\beta],H)$ we have

$$\begin{aligned}(w-P_V)^{-1} f(x) \quad &= \quad i2\pi M_w \int_\alpha^\beta e(w(x-s)) f(s)ds \\ &\qquad -i2\pi\int_\alpha^x e(w(x-s)) f(s)ds\end{aligned}$$

where $x\in[\alpha,\beta]$,

$$M_w := (1-e(-Lw)V)^{-1},$$

and $L:=\beta-\alpha$. Hence,

$$\left((\overline{z}-P_V)^{-1} - (z-P_V)^{-1}\right) f(x) = A+B$$

where

$$A := i2\pi M_{\overline{z}}\int_\alpha^\beta e(\overline{z}(x-s)) f(s)ds - i2\pi M_z \int_\alpha^\beta e(z(x-s)) f(s)ds$$

and

$$B := -i2\pi\int_\alpha^x \left(e(\overline{z}(x-s)) - e(z(x-s))\right) f(s)ds.$$

Now $|e(\overline{z}(x-s))-e(z(x-s))| = 2|\sin(b(x-s))| \leq 2bL$, so $B \to 0$ as $b \to 0$ uniformly in a. Consequently, when calculating

$$\tfrac{1}{2}\left(F[\mu,\nu]+F(\mu,\nu)\right)f(x)=\lim_{b\searrow 0}\int_{\mathbb{R}}\chi_{(\mu,\nu)}(a)Ada.$$

We may consider

$$i2\pi\left(M_{\overline{z}}-M_z\right)\int_{\alpha}^{\beta}e\left(z\left(x-s\right)\right)f(s)ds$$

in place of A because the norm of

$$\int_{\mu}^{\nu}i2\pi M_z\left(\int_{\alpha}^{\beta}e\left(z\left(x-s\right)\right)f(s)ds-\int_{\alpha}^{\beta}e\left(\overline{z}\left(x-s\right)\right)f(s)ds\right)da$$

is bounded above by

$$\int_{\mathbb{R}}\chi_{(\mu,\nu)}(a)\left\|M_z\right\|2bLda=\int_{\mathbb{R}}\chi_{(\mu,\nu)}(a)\left\|\left(1-e\left(-Lz\right)V\right)^{-1}\right\|2bLda\underset{b\searrow 0}{\longrightarrow}0.$$

The limit $=0$ because

$$\begin{aligned}\frac{b}{1-e\left(\lambda-L\left(a+ib\right)\right)} &= \frac{b}{1-e^{2\pi b}e\left(\lambda-La\right)}\\ &= -\frac{be^{-2\pi b}e\left(La-\lambda\right)}{1-e^{-2\pi b}e\left(La-\lambda\right)}\\ &= -b\sum_{m=1}^{\infty}e^{-2\pi mb}e_m\left(La-\lambda\right)\end{aligned}$$

and

$$\begin{aligned}&\left|\int_{\mu}^{\nu}-b\sum_{m=1}^{\infty}e^{-2\pi mb}e^{i2\pi(la-\lambda)m}da\right|\\ &=\left|-b\sum_{m=1}^{\infty}e^{-2\pi mb}e^{-i2\pi\lambda m}\tfrac{1}{i2\pi Lm}\left(e^{i2\pi mL\nu}-e^{i2\pi mL\nu}\right)\right|\\ &\leq\tfrac{b}{\pi L}\sum_{m=1}^{\infty}e^{-2\pi mb}\tfrac{1}{m}\\ &=\tfrac{b}{\pi L}\log\left(1-e^{-2\pi b}\right)\end{aligned}$$

which $\to 0$ (uniformly in) as $b\searrow 0$. Hence,

$$\begin{aligned}&\tfrac{1}{2}\left(F\left((\mu,\nu)\right)+F\left([\mu,\nu]\right)\right)f(x)\\ &=\lim_{b\searrow 0}\int_{\mathbb{R}}i2\pi\chi_{(\mu,\nu)}(a)\left(M_{\overline{z}}-M_z\right)\int_{\alpha}^{\beta}e\left(z\left(x-s\right)\right)f(s)dsda\end{aligned}\tag{4.17}$$

Using the spectral resolution (4.13) of V we have

$$\begin{aligned}M_{\overline{z}}-M_z&=\left(1-e\left(-L\overline{z}\right)V\right)^{-1}-\left(1-e\left(-Lz\right)V\right)^{-1}\\ &=\int_{[0,1)}\frac{1}{1-e(\lambda-La)e(iLb)}-\frac{1}{1-e(\lambda-La)e(-iLb)}E_V(d\lambda)\\ &=\int_{[0,1)}Q\left(r_b,\lambda-La\right)E_V(d\lambda),\end{aligned}$$

where $r_b := e(-iLb) = e^{2\pi Lb}$ and

$$Q(r,\theta) := \frac{1-r^2}{1-2r\cos(2\pi\theta)+r^2}$$

is the Poisson kernel for the unit circle. Consequently,

$$\begin{aligned}
&\tfrac{1}{2}\left(F\left((\mu,\nu)\right)+F\left([\mu,\nu]\right)\right)f(x)\\
&=\lim_{b\searrow 0}\int_{\mathbb{R}} i2\pi\chi_{(\mu,\nu)}(a)\left(\int_{[0,1)}Q\left(r_b,\lambda-La\right)E_V(d\lambda)\right)\int_\alpha^\beta e\left(z\left(x-s\right)\right)f(s)dsda\\
&=\lim_{b\searrow 0}\int_{[0,1)}\int_0^{\frac{1}{L}}\left(\sum_{m\in\mathbb{Z}}\chi_{(\mu,\nu)}\left(\tfrac{m}{L}+a\right)\right)Q\left(r_b,\lambda-La\right)E_V\left(d\lambda\right)e(zx)\widehat{f}(z)da\\
&=\int_{[0,1)}\left(\tfrac{1}{L}\sum_{m\in\mathbb{Z}}\chi_{(\mu,\nu)}\left(\tfrac{m+\lambda}{L}\right)\right)E_V\left(d\lambda\right)e\left(\tfrac{\lambda}{L}x\right)\widehat{f}\left(\tfrac{\lambda}{L}\right).
\end{aligned}$$

Where $\widehat{f}(z) := \int_\alpha^\beta e\left(-zs\right)f(s)ds$. The last equality follows from the Poisson kernel $Q(r,\theta)$ being an approximate identity as $r \nearrow 1$. ☐

Theorem 1.3 is a special case of the following corollary to Theorem 4.3:

Corollary 4.4. *The spectrum of P_V equals the set of $\lambda\in\mathbb{R}$ for which $e\left((\beta-\alpha)\lambda\right)$ is in the spectrum of V.*

Proof. By (4.13) the support of E_V is the set $\{\lambda\in[0,1] \mid e(\lambda)\in\text{spectrum}(V)\}$. Hence the result follows from (4.15). ☐

Example 4.5. If the spectrum of V equals the unit circle, then the spectrum of P_V equals the real line. In particular, the spectrum of P_V need not be discrete.

Theorem 4.6. *Let P_V be the selfadjoint extension of P_0, in $\mathscr{H} := L^2\left([\alpha,\beta]\right)\otimes H$, associated with the boundary unitary operator $V : H\to H$ by (1.2) and (1.3). Then $P_V + 1/\left(\beta-\alpha\right)$ is unitarily equivalent to P_V.*

Proof. For all $f\in\mathscr{H}$, define

$$\hat{f}\left(\lambda+m,y\right) := \frac{1}{\beta-\alpha}\int_\alpha^\beta \overline{e\left(\tfrac{\lambda+m}{\beta-\alpha}t\right)}f(t,y)dt;$$

See (4.14). By Theorem 4.3, we have

$$f\left(x,y\right) = \int_{[0,1)}\sum_{m\in\mathbb{Z}} e_{\frac{\lambda+m}{\beta-\alpha}}\left(x\right)\otimes E_V\left(d\lambda\right)\hat{f}\left(\lambda+m,y\right).$$

Moreover, for all $s\in\mathbb{R}$,

$$e\left(sP_V\right)f\left(x,y\right) = \int_{[0,1)}\sum_{m\in\mathbb{Z}} e\left(\tfrac{\lambda+m}{\beta-\alpha}s\right)e_{\frac{\lambda+m}{\beta-\alpha}}\left(x\right)\otimes E_V\left(d\lambda\right)\hat{f}\left(\lambda+m,y\right).$$

Let U be the unitary determined by

$$Uf\left(x,y\right) = \int_{[0,1)}\sum_{m\in\mathbb{Z}} e_{\frac{\lambda+m+1}{\beta-\alpha}}\left(x\right)\otimes E_V\left(d\lambda\right)\hat{f}\left(\lambda+m,y\right).$$

A direct computation shows that

$$e\left(sP_V\right)Uf = Ue\left(s\left(P_V+\frac{1}{\beta-\alpha}\right)\right)f.$$

If, in addition, $f \in \operatorname{dom}(P_V)$, then differentiating the last equation at $s = 0$ yields

$$P_V U f = U \left(P_V + \frac{1}{\beta - \alpha} \right) f.$$

That is, $P_V U = U (P_V + 1/(\beta - \alpha))$. This proves the theorem. $\square$

For simplicity suppose $\beta - \alpha = 1$. The previous two results suggests that P_V is unitary equivalent to $\bigoplus_{k\in\mathbb{Z}} (L + k)$ for some bounded selfadjoint operator $0 \le L \le 1$. Establishing this as a consequence of Theorem 4.3 provides an alternative proof of Theorem 1.2 and of Theorem 1.3.

Theorem 4.7. *Suppose $\alpha = 0$ and $\beta = 1$, then $\mathscr{H} = L^2[0,1] \otimes L^2[0,1]$. Let P_V be the selfadjoint extension of P_0, in $\mathscr{H}$, associated with the boundary unitary operator $V : H \to H$ by (1.2) and (1.3),*

$$L_k := P_V E_{P_V}([k, k+1)),$$

and $H_k := E_{P_V}([k, k+1))\mathscr{H}$. Clearly, L_k is a selfadjoint operator acting in H_k, with spectrum contained in $[k, k+1]$ and $k+1$ is not an eigenvalue of L_k. Furthermore, P_V is unitary equivalent to $\bigoplus_{k\in\mathbb{Z}} (L_0 + k)$.

Proof. Clearly, $P_V = \bigoplus_{k\in\mathbb{Z}} L_k$. For all $f \in \mathscr{H}$, and $k \in \mathbb{Z}$, define

$$\hat{f}(\lambda + k, y) := \int_0^1 \overline{e_{\lambda+k}(t)} f(t, y)\, dt.$$

By Theorem 4.3, we have

$$E_{P_V}([k, k+1)) f(x, y) = \int_{[0,1)} e_{\lambda+k}(x)\, E_V(d\lambda)\, \hat{f}(\lambda + k, y);$$

and

$$P_V E_{P_V}([k, k+1)) f(x, y) = \int_{[0,1)} (\lambda + k)\, e_{\lambda+k}(x)\, E_V(d\lambda)\, \hat{f}(\lambda + k, y).$$

Let $U_k : H_k \to H_0$ be the unitary operator determined by

$$\begin{aligned} U_k E_{P_V}([k, k+1)) f(x, y) &= U_k \left(\int_{[0,1)} e_{\lambda+k}(x)\, E_V(d\lambda)\, \hat{f}(\lambda + k, y) \right) \\ &= \int_{[0,1)} e_\lambda(x)\, E_V(d\lambda)\, \hat{f}(\lambda + k, y). \end{aligned}$$

A direct computation shows that

$$(P_V + k) U_k E_{P_V}([k, k+1)) f = U_k P_V E_{P_V}([k, k+1)) f,$$

i.e.,

$$(P_V + k) U_k = U_k P_V E_{P_V}([k, k+1)).$$

we then get

$$(L_0 + k) U_k = U_k L_k.$$

Notice that $P_V U_k = P_V E_{P_V}([0, 1)) U_k$.

Let $U := \bigoplus_{k\in\mathbb{Z}} U_k$, and it follows that

$$U P_V U^* = U \left(\bigoplus_{k\in\mathbb{Z}} L_k \right) U^* = \bigoplus_{k\in\mathbb{Z}} U_k L_k U_k^* = \bigoplus_{k\in\mathbb{Z}} (L_0 + k).$$

This completes the proof. $\square$

Remark 4.8. The analogue of Theorem 4.7 for P_θ in $L^2([0,1])$, $0 \leq \theta < 1$, determined by $P_\theta f = (i2\pi)^{-1} f'$ and $f(1) = e(\theta) f(0)$, states that P_θ is unitary equivalent to $\bigoplus (L_0 + k)$ in $\ell^2 = \bigoplus H_0$, where $H_k = \mathbb{C}$ for all k, and $L_k z = (\theta + k) z$ for $z \in \mathbb{C} = H_k$.

5. Spectral Pairs

In this section we consider momentum operators on product domains $[0,1] \times \Omega$ in $\mathbb{R}^2$, in the cases where $\Omega = \mathbb{R}$, $\Omega = [0,1]$, and where Ω is a certain fractal. We investigate when the momentum operators in the x and y directions commute in terms of the boundary unitaries.

Recall, two (unbounded) selfadjoint operators A and B commute if and only if their spectral measures commute. This is equivalent to the commutation of the unitary one-parameter groups $e(aA)$ and $e(bB)$ in the sense that

$$e(aA)e(bB) = e(bB)e(aA)$$

for all a, b in $\mathbb{R}$. See, e.g., [RS72].

5.1. The Infinite Strip

In this section we consider the infinite strip $[0,1] \times \mathbb{R}$. We obtain a complete classification of the commuting selfadjoint extensions of $(i2\pi)^{-1} \partial_x$ and $(i2\pi)^{-1} \partial_y$ acting in $C_c^\infty([0,1] \times \mathbb{R})$ in terms of the boundary unitary associated with $(i2\pi)^{-1} \partial_x$. Our method yields a complete list of the spectra of the infinite strip. This set was shown to be a spectral set in [Ped87], but the approach there only yields a partial list of the possible spectra.

Theorem 5.1. *Let $\mathscr{H} := L^2\left([\alpha,\beta], L^2(\mathbb{R})\right)$. Suppose $P = P_U := \frac{1}{i2\pi} \frac{\partial}{\partial x}\Big|_{\mathrm{dom}(P_U)}$ is the selfadjoint extension corresponding to the unitary operator $U : L^2(\mathbb{R}) \to L^2(\mathbb{R})$. Define $Q := \frac{1}{i2\pi} \frac{\partial}{\partial y}\Big|_{\mathrm{dom}(Q)}$, whose domain* $\mathrm{dom}(Q)$ *consists of all $f \in \mathscr{H}$, such that $\frac{\partial}{\partial y} f$ (in the sense of distribution) is in $\mathscr{H}$. Then P and Q commute if and only if U is diagonalized via Fourier transform, as*

$$U = \int_{\mathbb{R}} e(\gamma(\lambda)) \, |e_\lambda\rangle\langle e_\lambda| \, d\lambda \tag{5.1}$$

where $\gamma : \mathbb{R} \to [0,1)$ is a Borel function.

Proof. Note that Q is selfadjoint, and the unitary one-parameter group $e(tQ)$, $t \in \mathbb{R}$, is given by

$$e(tQ) f(x,y) = f(x, y+t)$$

for all $f \in \mathscr{H}$. That is,

$$e(tQ) = I \otimes \tau_t \tag{5.2}$$

where τ_t is the translation group in $L^2(\mathbb{R})$. Further, Q is diagonalized via the Fourier transform

$$Q = I \otimes \int_{-\infty}^{\infty} \lambda \, |e_\lambda\rangle\langle e_\lambda| \, d\lambda. \tag{5.3}$$

Here, $d\lambda$ denotes the Lebesgue measure on $\mathbb{R}$.

Now, suppose the two unitary one-parameter groups commute. By Lemma 2.2,

$$e((\beta - \alpha) P) = I \otimes U.$$

It follows that $I\otimes U$ commutes with $e\left(tQ\right)$, for all $t\in\mathbb{R}$. From (5.2)-(5.3), we see that

$$\begin{aligned}\left(I\otimes U\right)e\left(tQ\right)&=\left(I\otimes U\right)\left(I\otimes\tau_{t}\right)=I\otimes U\tau_{t}\\ e\left(tQ\right)\left(I\otimes U\right)&=\left(I\otimes\tau_{t}\right)\left(I\otimes U\right)=I\otimes\tau_{t}U;\end{aligned}$$

hence $U\tau_{t}=\tau_{t}U$, for all $t\in\mathbb{R}$. Consequently, e.g., [SW71, Theorem 3.16], U is diagonalized via the Fourier transform, as in (5.1).

Conversely, suppose U is given by (5.1). Fix $f\in\mathscr{H}$, $t\in\mathbb{R}$. For all $n\in\mathbb{Z}$, we have

$$\left\langle f,e\left(tQ\right)\left(1\otimes U^{n}\right)f\right\rangle =\int_{[0,1)}e_{n}\left(\lambda\right)\left\langle f,e\left(tQ\right)E_{U}\left(d\lambda\right)f\right\rangle ; \tag{5.4}$$

and

$$\left\langle f,\left(1\otimes U^{n}\right)e\left(tQ\right)f\right\rangle =\int_{[0,1)}e_{n}\left(\lambda\right)\left\langle f,E_{U}\left(d\lambda\right)e\left(tQ\right)f\right\rangle . \tag{5.5}$$

Note that, by assumption, U is diagonalized via Fourier transform, and so $1\otimes U^{n}$ commutes with $e\left(tQ\right)$, for all $n\in\mathbb{Z}$. Therefore, the two Borel measures, on the right-hand-side of (5.4) and (5.5), have the same Fourier coefficients; thus

$$\left\langle f,e\left(tQ\right)E_{U}\left(d\lambda\right)f\right\rangle =\left\langle f,E_{U}\left(d\lambda\right)e\left(tQ\right)f\right\rangle . \tag{5.6}$$

Multiplying $e\left(s\lambda\right)$ on both sides of (5.6) and integrating over $[0,1)$, we get

$$\int_{[0,1)}e\left(s\lambda\right)\left\langle f,e\left(tQ\right)E_{U}\left(d\lambda\right)f\right\rangle =\int_{[0,1)}e\left(s\lambda\right)\left\langle f,E_{U}\left(d\lambda\right)e\left(tQ\right)f\right\rangle$$

i.e.,

$$\left\langle f,e\left(tQ\right)e\left(sP\right)f\right\rangle =\left\langle f,e\left(sP\right)e\left(tQ\right)f\right\rangle$$

for all $s\in\mathbb{R}$.

Since f and t are arbitrary, we conclude that $e\left(sP\right)$ commutes with $e\left(tQ\right)$, for all $s,t\in\mathbb{R}$. ☐

Remark 5.2. To put U in (5.1) into the standard projection-valued measure form (4.13), let $E\left(d\lambda\right):=\left|e_{\lambda}\right\rangle \left\langle e_{\lambda}\right|d\lambda$, and $E_{U}:=E\circ\gamma^{-1}$. Hence,

$$U=\int_{\mathbb{R}}e\left(\gamma\left(\lambda\right)\right)E\left(d\lambda\right)=\int_{[0,1)}e\left(\lambda\right)E\left(\gamma^{-1}\left(d\lambda\right)\right)=\int_{[0,1)}e\left(\lambda\right)E_{U}\left(d\lambda\right).$$

Note that, for all $\varphi\in L^{2}\left(\mathbb{R}\right)$, and all Borel set $\triangle$ in $\mathbb{R}$,

$$\begin{aligned}\left\Vert E\left(\triangle\right)\varphi\right\Vert ^{2}&=\int_{\triangle}\left|\widehat{\varphi}\left(\lambda\right)\right|^{2}d\lambda\\ \left\Vert E_{U}\left(\triangle\right)\varphi\right\Vert ^{2}&=\int_{\gamma^{-1}\left(\triangle\right)}\left|\widehat{\varphi}\left(\lambda\right)\right|^{2}d\lambda.\end{aligned}$$

Remark 5.3. It follows from Theorem 4.3 and Remark 5.2 that

$$\begin{aligned}P_{U}&=\int_{[0,1)}\sum_{m\in\mathbb{Z}}\left(\lambda+m\right)\left|e_{\lambda+m}\right\rangle \left\langle e_{\lambda+m}\right|\otimes E_{U}\left(d\lambda\right)\\ &=\int_{\mathbb{R}}\sum_{m\in\mathbb{Z}}\left(\gamma(\lambda)+m\right)\left|e_{\gamma(\lambda)+m}\right\rangle \left\langle e_{\gamma(\lambda)+m}\right|\otimes\left|e_{\lambda}\right\rangle \left\langle e_{\lambda}\right|d\lambda\end{aligned}$$

Moreover, Q is diagonalized via the Fourier transform, see (5.3). Therefore, the joint spectrum of P_U and Q is the closure of the set

$$\Lambda_\gamma := \left\{ \begin{pmatrix} \gamma(\lambda)+m \\ \lambda \end{pmatrix} \Big| \, m \in \mathbb{Z}, \lambda \in \mathbb{R} \right\},$$

provided γ has been chosen such that $e(\gamma(\lambda))$ is in the spectrum of U for all λ.

5.2. **The Unit Square**

In this section we consider the unit square $[0,1]^2$. We obtain a complete classification of the commuting extensions of $\frac{1}{i2\pi}\partial_x$ and $\frac{1}{i2\pi}\partial_y$ acting in $C_c^\infty\left([0,1]^2\right)$ in term of the boundary unitaries, see also [JP00]. As a consequence we recover the list of all possible spectra of $[0,1]^2$ first obtained in [JP99].

Lemma 5.4. *Let $(X, \mathfrak{M}_X, \mu)$ and $(Y, \mathfrak{M}_Y, \nu)$ be measure spaces, where μ is a complex measure on $\mathfrak{M}_X$ and ν a positive measure on $\mathfrak{M}_Y$. Let $\pi : X \to Y$ be a measurable function.*

Suppose there is a family of measures $\{\psi(y,\cdot)\}_{y\in Y}$, such that,

1. *For all $y \in Y$, $\psi(y,\cdot)$ is supported in $\pi^{-1}(y)$;*
2. *For all $B \in \mathfrak{M}_X$, $\psi(\cdot, B) \in L^1(d\nu)$; and*

$$\mu(B) = \int \psi(y,B)\,\nu(dy). \tag{5.7}$$

Then, for each $B \in \mathfrak{M}_X$, $\psi(\cdot,B)$ is uniquely determined. That is, if $\{\psi'(y,\cdot)\}_{y\in Y}$ is another family of measures satisfying (1)-(2), then for all $B \in \mathfrak{M}_X$,

$$\psi(\cdot,B) = \psi'(\cdot,B), \quad \nu \text{ - a.e.}$$

Proof. Fix $B \in \mathfrak{M}_X$. For all $F \in \mathfrak{M}_Y$, we have

$$\int_F \psi(y,B)\,\nu(dy) = \int \psi\left(y, B \cap \pi^{-1}(F)\right)\nu(dy) = \mu\left(B \cap \pi^{-1}(F)\right). \tag{5.8}$$

Note the first equality follows from the assumption that $\psi(y,\cdot)$ is supported in $\pi^{-1}(y)$, for all $y \in Y$.

Consequently, $\mu\left(B \cap \pi^{-1}(\cdot)\right) \ll \nu$, and the associated Radon-Nikodym derivative is $\psi(\cdot,B)$. If $\{\psi'(y,\cdot)\}_{y\in Y}$ is another family of measures as stated, then (5.8) holds with ψ' on the left-hand-side. The uniqueness of Radon-Nikodym derivative then implies that $\psi(\cdot,B) = \psi'(\cdot,B)$, ν-a.e. □

Theorem 5.5. *Let $\mathscr{H} := L^2[0,1] \otimes L^2[0,1]$. Let $P = P_U := \frac{1}{i2\pi}\frac{\partial}{\partial x}\Big|_{\mathrm{dom}(P_U)}$ and $Q = Q_V := \frac{1}{i2\pi}\frac{\partial}{\partial y}\Big|_{\mathrm{dom}(Q)}$ be the selfadjoint extensions corresponding to the boundary unitary operators $U : L^2(I_y) \to L^2(I_y)$, $V : L^2(I_x) \to L^2(I_x)$, respectively.*

Then P and Q commute if and only if there are $\alpha, \beta_m \in [0,1)$ such that

$$Ve_{\alpha+m} = e(\beta_m)e_{\alpha+m} \text{ and } U = e(\alpha)I \tag{5.9}$$

or

$$V = e(\alpha)I \text{ and } Ue_{\alpha+m} = e(\beta_m)e_{\alpha+m} \tag{5.10}$$

for all $m \in \mathbb{Z}$.

Proof. Suppose

$$U = \int_{[0,1)} e(\lambda)\, E_U(d\lambda) \tag{5.11}$$

$$P = \int_{\mathbb{R}} \lambda E_P(d\lambda) \tag{5.12}$$

where E_U and E_P are the respective projection-valued measures. By Theorem 4.3, for all Borel set $\triangle \subset \mathbb{R}$,

$$E_P(\triangle) = \int_{[0,1)} \Psi(\lambda, \triangle) \otimes E_U(d\lambda)\,; \text{ where} \tag{5.13}$$

$$\Psi(\lambda, \triangle) := \sum_{m\in\mathbb{Z}} \chi_{\triangle}(\lambda+m)\, |e_{\lambda+m}\rangle\langle e_{\lambda+m}|\,. \tag{5.14}$$

Let $f \otimes g \in \mathscr{H}$, then

$$\langle f\otimes g, (V\otimes I)\, E_P(\triangle)\, f\otimes g\rangle = \int_{[0,1)} \langle f, V\Psi(\lambda,\triangle)\, f\rangle \left\|E_U(d\lambda)\, g\right\|^2 \tag{5.15}$$

$$\langle f\otimes g, E_P(\triangle)\,(V\otimes I)\, f\otimes g\rangle = \int_{[0,1)} \langle f, \Psi(\lambda,\triangle)\, Vf\rangle \left\|E_U(d\lambda)\, g\right\|^2. \tag{5.16}$$

Now, suppose $e(sP)$ and $e(tQ)$ commute, for all $s, t \in \mathbb{R}$. In particular, by Lemma 2.2

$$V \otimes I = e(Q)$$

so that $V\otimes I$ commutes with $e(sP)$, $s \in \mathbb{R}$. Similarly, $I \otimes U$ commutes with $e(tQ)$, $t\in\mathbb{R}$.

Hence, the two complex Borel measures on the left-hand-side of (5.15)-(5.16) are identical. We denote this measure by μ. Also, let

$$\nu(d\lambda) := \left\|E_U(d\lambda)\, g\right\|^2.$$

Define $\pi : (\mathbb{R}, \mu) \to (\mathbb{T} \cong [0,1), \nu)$ as the quotient map. Set

$$\begin{aligned} \psi_1(\lambda,\cdot) &:= \langle f, V\Psi(\lambda,\cdot)\, f\rangle \\ \psi_2(\lambda,\cdot) &:= \langle f, \Psi(\lambda,\cdot)\, Vf\rangle\,. \end{aligned}$$

Then, for $j = 1, 2$, we have

1. For all $\lambda \in [0,1)$, $\psi_j(\lambda,\cdot)$ is supported in $\pi^{-1}(\lambda) = \lambda + \mathbb{Z}$; see (5.14);
2. $\psi_j(\cdot,\triangle) \in L^\infty(\nu)$, and

   $$\mu(\triangle) = \int_{[0,1)} \psi_j(\lambda,\triangle)\, \nu(d\lambda)\,;$$

   see (5.15)-(5.16).

Thus, by Lemma 5.4, $\langle f, V\Psi(\lambda,\triangle)\, f\rangle = \langle f, \Psi(\lambda,\triangle)\, Vf\rangle$, ν-a.e. Since f is arbitrary, we conclude that

$$V\Psi(\lambda,\triangle) = \Psi(\lambda,\triangle)\, V, \quad \nu\text{ - }a.e. \tag{5.17}$$

for each Borel set $\triangle \subset \mathbb{R}$.

Note that $\Psi(\lambda,\cdot)$ is a resolution of identity in $L^2[0,1]$, thus (5.17) implies that there exists $\beta_{\lambda+m} \in [0,1)$, $m \in \mathbb{Z}$, such that

$$Ve_{\lambda+m} = e(\beta_{\lambda+m})\, e_{\lambda+m}, \quad \nu\text{ - }a.e. \tag{5.18}$$

Let S be the set of $\lambda \in [0,1)$ such that (5.17) holds, thus, $\nu(S^c) = 0$. We proceed to show there are two possibilities:

Case 1. $S = \{\alpha\}$, i.e., a singleton. Then (5.18) yields

$$V = \sum_{m\in\mathbb{Z}} e(\beta_m) \, |e_{\alpha+m}\rangle\langle e_{\alpha+m}| .$$

Moreover, since $\nu_g(d\lambda) = \|E_U(d\lambda) g\|^2$ is supported at $\{\alpha\}$ and g was arbitrary, it follows that

$$U = e(\alpha) I.$$

This yields (5.9).

Case 2. S consists of more than one point. Let λ, λ' be distinct points in S. Then

$$\begin{aligned} & (e(\beta_{\lambda+m}) - e(\beta_{\lambda'+m'})) \langle e_{\lambda'+m'}, e_{\lambda+m}\rangle \\ = \; & \langle e_{\lambda'+m'}, V e_{\lambda+m}\rangle - \langle V^* e_{\lambda'+m'}, e_{\lambda+m}\rangle \\ = \; & 0. \end{aligned}$$

Since $\langle e_{\lambda'+m'}, e_{\lambda+m}\rangle \neq 0$, there is a constant $\alpha \in [0,1)$, such that $\beta_{\lambda+m} = \alpha$, for all $\lambda \in [0,1)$, and $m \in \mathbb{Z}$. That is,

$$V = e(\alpha) I.$$

We then run through the argument used in the proof, starting with the fact that $I \otimes U$ commutes with $e(tQ)$, $t \in \mathbb{R}$. It follows that

$$U = \sum_{m\in\mathbb{Z}} e(\beta_n) \, |e_{\alpha+n}\rangle\langle e_{\alpha+n}| .$$

This yields (5.10).

The converse is essentially trivial. For example, if (5.9), then the functions $e_{\alpha+m} \otimes e_{\beta_m+n}$, $m, n \in \mathbb{Z}$ is a complete set of joint eigenfunctions for P_U and Q_V. $\square$

Remark 5.6. In case (5.9) the joint spectrum of P and Q is

$$\left\{ \begin{pmatrix} \alpha + m \\ \beta_m + n \end{pmatrix} \mid m, n \in \mathbb{Z} \right\}, \tag{5.19}$$

since Theorem 4.3 in this case states that

$$\begin{aligned} P &= \sum_{m\in\mathbb{Z}} (m+\alpha) \, |e_{\alpha+m}\rangle\langle e_{\alpha+m}| \otimes I \\ &= \sum_{m,n\in\mathbb{Z}} (m+\alpha) \, |e_{\alpha+m}\rangle\langle e_{\alpha+m}| \otimes \Big(|e_{\beta_m+n}\rangle\langle e_{\beta_m+n}| \Big) \\ Q &= \sum_{m,n\in\mathbb{Z}} (\beta_m + n) \Big(|e_{\alpha+m}\rangle\langle e_{\alpha+m}| \Big) \otimes \Big(|e_{\beta_m+n}\rangle\langle e_{\beta_m+n}| \Big). \end{aligned}$$

Similarly, in case (5.10) the joint spectrum is

$$\left\{ \begin{pmatrix} \beta_n + m \\ \alpha + n \end{pmatrix} \mid m, n \in \mathbb{Z} \right\}. \tag{5.20}$$

That this is the possible joint spectra was established in [JP00] by a different method.

Remark 5.7. Suppose (5.10), then U is unitary equivalent to $\widetilde{U}e_m = e(\beta_m)\, e_m$. Hence P_U is unitary equivalent to $P_{\widetilde{U}}$ by Theorem 2.3. Furthermore, $\widetilde{U}$ is a geometric boundary condition, more precisely, a rotation if and only if there is a real number r, such that β_m is the fractional part $\langle rm\rangle$ of rm for all m. See Remark 3.2 and Remark 3.5.

5.3. **A Fractal**

Let μ be a probability measure with support $C \subset \mathbb{R}$. Suppose the functions

$$e_\lambda, \lambda \in \Lambda$$

form an orthonormal basis for $L^2(\mu)$. Let Q be the selfadjoint operator determined by

$$Q\left(\sum_\lambda c_\lambda g_\lambda \otimes e_\lambda\right) = \sum_\lambda \lambda c_\lambda g_\lambda \otimes e_\lambda$$

whose domain is the set of all $g \in L^2([0,1])$ and all finite sums $\sum_\lambda c_\lambda g_\lambda \otimes e_\lambda$ with $c_\lambda \in \mathbb{C}$, $g_\lambda \in L^2([0,1])$, and $\lambda \in \Lambda$. Then Q is essentially selfadjoint and $Qf = \frac{1}{i2\pi}\partial_y f$ for any $f = g \otimes \sum_\lambda c_\lambda e_\lambda$. See also Appendix B. We also denote the closure of this operator by Q.

Theorem 5.8. *Let U be a unitary on $H = L^2(\mu)$ and let P_U be the corresponding selfadjoint extension of P_0, in $L^2([0,1]) \otimes H$ determined by (1.2) and (1.3). Then P_U and Q commute if and only if the function e_λ, $\lambda \in \Lambda$ are eigenfunctions for U.*

Proof. Since $e(tQ)f \otimes e_\lambda = e(t\lambda)f \otimes e_\lambda$ for all $f \in L^2([0,1])$ and all $\lambda \in \Lambda$ it follows that

$$e(tQ) = I \otimes e(t\widetilde{Q}) \tag{5.21}$$

where $\widetilde{Q}$ acting in $L^2(\mu)$ is determined by $\widetilde{Q}e_\lambda = \lambda e_\lambda$ for $\lambda \in \Lambda$.

By Lemma 2.2 $e(P) = I \otimes U$, so it follows from $e(tQ)e(P) = e(P)e(tQ)$ and (5.21) that

$$e(t\widetilde{Q})Ue_\lambda = e(t\lambda)Ue_\lambda.$$

Consequently, Ue_λ is a multiple of e_λ.

The converse is trivial, see the proof of Theorem 5.5. $\square$

Remark 5.9. If $\gamma : \Lambda \to [0,1)$ is such that $Ue_\lambda = e(\gamma(\lambda))e_\lambda$, then the functions

$$e_{\gamma(\lambda)+m} \otimes e_\lambda, m \in \mathbb{Z}, \lambda \in \Lambda$$

form an orthonormal basis for $L^2([0,1]) \otimes L^2(\mu)$ consisting of joint eigenfunctions for P_U and Q. Consequently, the joint spectrum of P_U and Q is the closure of the set of (joint) eigenvalues

$$\Lambda_\gamma := \left\{ \begin{pmatrix} \gamma(\lambda) + m \\ \lambda \end{pmatrix} \mid m \in \mathbb{Z}, \lambda \in \Lambda \right\}. \tag{5.22}$$

Example 5.10. Consider the Cantor set

$$C := \left\{ \sum_{k=1}^{\infty} d_k 4^{-k} \mid d_k \in \{0,3\} \right\}$$

and the set

$$\Lambda := \left\{ \sum_{k=0}^{n} d_k 4^{k} \mid d_k \in \{0,1\} \right\}.$$

If μ is the measure determined by

$$\mu\left(\left[\sum_{k=1}^{n} d_k 4^{-k}, \sum_{k=1}^{n} d_k 4^{-k} + 4^{-n}\right]\right) = 2^{-n}$$

for all $n \geq 1$, where $d_k \in \{0,3\}$, then C is the support of C and it was shown in [JP98] that the functions e_λ, $\lambda \in \Lambda$ form an orthonormal basis for $L^2(\mu)$. The set Λ is called a *spectrum* of μ. If $\nu = m \otimes \mu$, where m is Lebesgue measure on the interval $[0,1]$, then it follows from [JP99] that

$$(\nu, \Lambda_\gamma)$$

is a spectral pair, where Λ_γ is determined by (5.22). This, combined with Remark 5.9, gives an explicit formula for the possible joint spectra of commuting pairs P_U, Q in terms of the choice of a function γ and the spectrum Λ of μ. Not all exponential basis for $L^2(\mu)$ are know, see the paper [DHS09] and its references for constructions of other exponential basis for $L^2(\mu)$.

Appendix A. Selfadjoint Extensions

Fix real numbers $\alpha < \beta$ and a Hilbert space H. Consider the Hilbert space

$$\mathscr{H} := L^2([\alpha,\beta], H) = L^2([\alpha,\beta]) \otimes H$$

of L^2-functions $[\alpha,\beta] \to H$ equipped with the inner product

$$\langle f \mid g \rangle := \int_\alpha^\beta \langle f(x) \mid g(x) \rangle \, dx,$$

where $\langle f(x) \mid g(x) \rangle$ is the inner product in H. We will consider selfadjoint restrictions of the operator $P = P_{\max}$ determined by

$$Pf := \frac{1}{i2\pi}\frac{d}{dx} f = \frac{1}{i2\pi} f', \tag{A.1}$$

with the (maximal) domain

$$\operatorname{dom}(P) := \left\{ f \in L^2([\alpha,\beta], H) : f' \in L^2([\alpha,\beta], H) \right\}. \tag{A.2}$$

Let $P_{\min}$ be the restriction of P to the (minimal) domain $\operatorname{dom}(P_{\min}) := C_c^\infty([\alpha,\beta]) \otimes H$. Finally, let P_0 be the restriction of P to the domain

$$\operatorname{dom}(P_0) := \{ f \in \operatorname{dom}(P) \mid f(\alpha) = f(\beta) = 0 \}.$$

Integrations by parts shows that $\langle P_0 f \mid g \rangle = \langle f \mid P_0 g \rangle$ for all f, g in $\operatorname{dom}(P_0)$ and consequently also $\langle P_{\min} f \mid g \rangle = \langle f \mid P_{\min} g \rangle$ for all f, g in $\operatorname{dom}(P_{\min})$. Hence, P_0 and $P_{\min}$ are densely defined symmetric operators in $L^2([\alpha,\beta], H)$.

Clearly, $P_{\min}$ is a restriction of P_0. A consequence of the next lemma is that $P_{\min}$ and P_0 have the same selfadjoint extensions.

Lemma A.1. *We have*

$$P_{\min}^* = P_0^* = P.$$

Recall, $P = P_{\max}$.

Proof. Fix $f \in L^2([\alpha,\beta],H)$ with $f' \in L^2([\alpha,\beta],H)$. For $g \in C_c^\infty([\alpha,\beta]) \otimes H$ integration by parts yields

$$\begin{aligned}\langle P_{\min} g \mid f\rangle &= \tfrac{1}{i2\pi}\int_\alpha^\beta \langle g'(x) \mid f(x)\rangle\, dx \\ &= -\tfrac{1}{i2\pi}\int_\alpha^\beta \langle g(x) \mid f'(x)\rangle\, dx,\end{aligned}$$

since $g(\alpha)=g(\beta)=0$. Consequently, f is in $\operatorname{dom}(P_{\min}^*)$ and $P_{\min}^* f = \frac{1}{i2\pi} f'$.

Conversely, fix $f \in D(P_{\min}^*)$. Let $g := P_{\min}^* f$ and $G(x) := \int_\alpha^x g(t)dt$. For $h \in C_c^\infty([\alpha,\beta) \otimes H$ we have $\langle P_{\min} h \mid f\rangle = \langle h \mid P_{\min}^* f\rangle = \langle h \mid g\rangle$, hence integration by parts leads to

$$\begin{aligned}\int_\alpha^\beta \tfrac{1}{i2\pi}\langle h'(x) \mid f(x)\rangle\, dx &= \int_\alpha^\beta \langle h(x) \mid g(x)\rangle\, dx \\ &= -\int_\alpha^\beta \langle h'(x) \mid G(x)\rangle\, dx\end{aligned}$$

since $h(\alpha)=h(\beta)=0$. Consequently,

$$\int_\alpha^\beta \left\langle h'(x) \,\middle|\, \frac{-1}{i2\pi} f(x) + G(x)\right\rangle dx = 0,$$

for all $h \in C_c^\infty([\alpha,\beta]) \otimes H$. It follows that $-\frac{1}{i2\pi} f(x) + G(x)$ is constant. Using the definition of G we conclude f' exists and

$$\frac{1}{i2\pi} f' = G' = g = P_0^* f.$$

Hence f is in $\operatorname{dom}(P)$ and $P_{\min}^* f = Pf$.

Repeating this argument shows that $P_0^* = P$. □

When working with the von Neumann parametrization of the selfadjoint extensions of a symmetric operator, it is important to start with a closed operator, hence the following lemma is important.

Lemma A.2. *The closure $\overline{P_{\min}}$ of $P_{\min}$ equals P_0, in particular, P_0 is closed.*

Proof. Using Lemma A.1 we see that $\overline{P_{\min}} = P_{\min}^{**} = P^* = P_0^{**} = \overline{P_0}$. Hence, it is sufficient to show that $P_0^{**} = P_0$. Fix $f \in \operatorname{dom}(P_0^{**})$. We must show $f \in \operatorname{dom}(P_0)$ and $P_0^{**} f = \frac{1}{i2\pi} f'$. Let $g := P_0^{**} f$ and $G(x) := \int_\alpha^x g(t)dt$. For $h \in \operatorname{dom}(P_0^*)$ we have $\langle P_0^* h \mid f\rangle = \langle h \mid P_0^{**} f\rangle = \langle h \mid g\rangle$. Hence integration by parts leads to

$$\begin{aligned}\int_\alpha^\beta \frac{1}{i2\pi}\langle h'(x) \,|\, f(x)\,\rangle\, dx &= \int_\alpha^\beta \langle h(x) \,|\, g(x)\,\rangle\, dx \\ &= B(h,G) - \int_\alpha^\beta \langle h'(x) \,|\, G(x)\,\rangle\, dx,\end{aligned}$$

where

$$B(h,G) := \langle h(\beta) \mid G(\beta)\rangle - \langle h(\alpha) \mid G(\alpha)\rangle\,.$$

We can add a constant function $\phi(x) \equiv \psi$ to h without changing h'. Hence, for such ϕ,

$$B(h,G) = B(h+\phi,G) = B(h,G) + B(\phi,G).$$

Consequently $B(\phi, G) = 0$ for all constant functions $\phi(x) \equiv \psi$. This means that

$$\langle \psi \mid G(\beta) - G)\alpha) \rangle = 0$$

for all $\psi \in H$. So $G(\beta) = G(\alpha) = 0$. Now it follows, as in the proof of the previous lemma, that $\frac{1}{i2\pi} f(x) - G(x)$ is constant in the x variable, hence f' exists and

$$\frac{1}{i2\pi} f' = G' = g = P_0^{**} f.$$

In particular, $f' \in L^2([\alpha, \beta], H)$ and $P_0^{**} f = \frac{1}{i2\pi} f'$.

It remains to check that $f(\alpha) = f(\beta) = 0$. If $g \in \operatorname{dom}(P_0^*)$, then $\langle P_0^* g \mid f \rangle = \langle g \mid P_0^{**} f \rangle$. This means that

$$\int_\alpha^\beta \langle g'(x) \mid f(x) \rangle \, dx = -\int_\alpha^\beta \langle g(x) \mid f'(x) \rangle \, dx.$$

Integration by parts shows that the two sides of this equation differ by

$$B(f, g) = \langle g(\beta) \mid f(\beta) \rangle - \langle g(\alpha) \mid f(\alpha) \rangle.$$

Hence, $B(f, g) = 0$ for all $g \in \operatorname{dom}(P^*)$. Since, $\beta \neq \alpha$, there are functions g in $\operatorname{dom}(P^*)$ that are zero on one boundary point and an arbitrary element of H on the other boundary point. Consequently, $f(\alpha) = f(\beta) = 0$. □

Lemma A.3. *The orthogonal complement of the range of P_0 is the set of functions f in $L^2(A)$ of the form $f(x, y) = h(y)$ for some h in H. The orthogonal complement of the range of $P_0 \pm i$ is the set of all functions $f_\pm$ in $L^2([\alpha, \beta], H)$ such that $f_\pm(x) = \exp(\pm 2\pi x) h$, for x in $[\alpha, \beta]$ and h in H.*

Proof. Suppose f is in the orthogonal complement to the range of P_0. Then

$$\langle P_0 g \mid f \rangle = 0$$

for all $g \in D(P_0)$, hence $f \in D(P_0^*)$ and $P_0^* f = 0$. By Lemma A.1 $f' = 0$. Solving this differential equation gives the desired conclusion. The calculation of the orthogonal complement of the range of $P_0 \pm i$ is similar. □

Proposition A.4. *The selfadjoint extensions of P_0 are parametrized by the unitaries $V : H \to H$. The selfadjoint extension P_V of P_0 corresponding to the unitary V is the restriction of $P = P_{\max}$ whose domain $\operatorname{dom}(P_V)$ consists of the functions*

$$f(x) + e^{2\pi(\beta - x)} h + e^{2\pi(x - \alpha)} V h,$$

where $f \in \operatorname{dom}(P_0)$ and $h \in H$. The action of P_V is

$$P_V \left(f(x) + e^{2\pi(\beta - x)} h + e^{2\pi(x - \alpha)} V h \right) = \frac{1}{i2\pi} f'(x) + i e^{2\pi(\beta - x)} h - i e^{2\pi(x - \alpha)} V h$$

where f and h are as above.

Proof. This is an application of the von Neumann index theory, see e.g., [RS75] for an account of this theory. P_0 is densely defined, since $C_c^\infty([\alpha, \beta]) \otimes H \subseteq \operatorname{dom}(P_0)$ and P_0 is closed by Lemma A.2. The deficiency spaces $\mathscr{D}_\pm(P_0) := \ker(P_0 \mp iI)$ of P_0 are

$$\mathscr{D}_\pm(P_0) = \left\{ f \in L^2([\alpha, \beta], H) \mid f(x) = \exp(\mp 2\pi x) \, h \, , h \in H \right\} \tag{A.3}$$

according to Lemma A.3. In particular, $\dim \mathscr{D}_+(P_0) = \infty = \dim \mathscr{D}_-(P_0)$, and consequently, P_0 has selfadjoint extensions.

By the von Neumann theory, the selfadjoint extensions of P_0 are parametrized by the partial isometries W with initial space $\mathscr{D}_+(P_0)$ and final space $\mathscr{D}_-(P_0)$. Specitically, the selfadjoint extension P_W corresponding to the partial isometry W is the restriction of $P = P_0^*$ to the domain

$$\operatorname{dom}(P_W) := \{f + f_+ + Wf_+ \mid f \in \operatorname{dom}(P_0), f_+ \in \mathscr{D}_+(P_0)\}.$$

If $f_+(x) := e^{2\pi(\beta-x)}h$ and $f_-(x) := e^{2\pi(x-\alpha)}Vh$, then

$$4\pi\|f_+\|^2_{L^2([\alpha,\beta],H)} = \left(e^{4\pi(\beta-\alpha)} - 1\right)\|h\|^2_H = 4\pi\|Vh\|^2_H.$$

The correspondance between W and V is determined by

$$We^{2\pi(\beta-x)}h = e^{2\pi(x-\alpha)}Vh.$$

The claims are now immediate. □

Another way to describe the selfadjoint extensions of P are through boundary conditions.

Proposition A.5. *The selfadjoint extensions of P_0 are parametrized by the unitaries $V : H \to H$. The selfadjoint extension P_V of P_0 corresponding to the unitary V is the restriction of $P = P_{\max}$ with domain*

$$\operatorname{dom}(P_V) := \{f \in \operatorname{dom}(P) \mid f(\beta) = Vf(\alpha)\}. \tag{A.4}$$

Proof. If $f \in \operatorname{dom}(P)$, then we saw in the proof of Lemma A.1 that

$$f(x) = h + \int_{\alpha(y)}^{x} Pf(t)dt,$$

for some h in H. In particular, $f(\alpha)$ and $f(\beta)$ are well-defined elements of H. Integration by parts shows that for $f, g \in \operatorname{dom}(P)$ we have

$$\langle Pf \mid g\rangle = B(f,g) + \langle f \mid Pg\rangle$$

where

$$B(f,g) := \frac{1}{i2\pi}\langle f(\beta) \mid g(\beta)\rangle - \langle f(\alpha) \mid g(\alpha)\rangle.$$

Since h is arbitrary in H, the maps

$$f \in \operatorname{dom}(P) \to f(\alpha) \in H \text{ and } f \in \operatorname{dom}(P) \to f(\beta) \in H$$

have dense ranges. Consequently, the result follows from [dO09, Theorem 7.1.13]. □

Let V_{vN} be the unitary from Proposition A.4 and let V_B be the unitary from Proposition A.5.

The function

$$g(x) = f(x) + e^{2\pi(\beta-x)}h + e^{2\pi(x-\alpha)}V_{vN}h,$$

from Proposition A.4 has boundary values $f(\alpha) = e^{2\pi(\beta-\alpha)}h + V_{nN}h$ and $f(\beta) = h + e^{2\pi(\beta-\alpha)}V_{vN}h$. Hence, $h = \left(e^{2\pi(\beta-\alpha)} + V_{vN}\right)^{-1} f(\alpha)$ and $f(\beta) = \left(1 + e^{2\pi(\beta-\alpha)}V_{vN}\right)h$. It follows that

$$V_B = \left(1 + e^{2\pi(\beta-\alpha)}V_{vN}\right)\left(e^{2\pi(\beta-\alpha)} + V_{vN}\right)^{-1}.$$

Conversely,

$$V_{vN} = \left(e^{2\pi(\beta-\alpha)} - V_B\right)^{-1}\left(e^{2\pi(\beta-\alpha)}V_B - 1\right).$$

Hence, Proposition A.4 and Proposition A.5 are, in fact, equivalent. In particular, we could have established Proposition A.5 without appealing to [dO09, Theorem 7.1.13].

Appendix B. Questions

Problem B.1. A set Λ is a tiling set for the square $[0,1]^2$ if $\bigcup_{\lambda\in\Lambda}\left(\lambda+[0,1]^2\right)=\mathbb{R}^2$ and the overlaps are null sets. It is known that Λ is the joint spectrum for some commuting extensions P, Q as in Theorem 5.5 if and only if Λ is a tiling set for the square, see [JP99], [IP98], [LRW00]. In case (5.9) with $\alpha=0$ and $\beta_m=\langle rm\rangle$ for some $r\in\mathbb{R}$ we see that both U and V are determined by the geometric boundary conditions from Remark 3.2. I might be of interest to investigate the relationship between geometric boundary conditions and the boundary unitary operators U and V in more detail.

Problem B.2. Suppose (μ,ν) is a spectral pair. Then $f(x)=\int\widehat{f}(\lambda)e(\lambda x)d\nu(\lambda)$. Let

$$e\left(bQ\right)f(x):=\int e\left(b\lambda\right)\widehat{f}(\lambda)e(\lambda x)d\nu(\lambda). \tag{B.1}$$

Then $e\left(bQ\right)f(x)=f(x+b)$ for a.e. x such that $x+b$ is in the support of μ. So, ignoring null sets, if $b_n\to 0$ and $x+b_n$ is in the support of μ for all n, then

$$\frac{e\left(b_nQ\right)f(x)-f(x)}{b_n}\to f'(x) \tag{B.2}$$

but the limit also equals $i2\pi Qf(x)$. Hence, Q determined by ($B.1$) can be thought of as a selfadjoint realization of $\frac{1}{i2\pi}\frac{d}{dx}$ in $L^2(\mu)$. Theorem 5.1 considers the case where μ is Lebesgue measure on the real line and Theorem 5.8 a certain restriction of $1/2-$dimensional Hausdorff measure. A common generalization of these two cases is:

Theorem. *Suppose (μ,ν) is a spectral pair of measures on $\mathbb{R}$ and Q is determined by ($B.1$). Let $\mathscr{H}:=L^2\left([0,1]\right)\otimes L^2(\mu)$, let U be a boundary unitary in $L^2\left(\mu\right)$ and let P_U be the corresponding selfadjoint extension of P_0. Then P_U and $I\otimes Q$ commute if and only if*

$$Uf(x)=\int e(\gamma(\lambda))\widehat{f}(\lambda)e(\lambda x)d\nu(\lambda)$$

for some $\nu-$measurable function $\gamma:\mathbb{R}\to[0,1)$.

Is there a way to generalize this to also include Theorem 5.5 as a special case?

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